# Quiz 4

December 6, 2018

Let

$$A = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}.$$

Determine whether $A$ is diagonalizable. If it is diagonalizable, find a nonsingular matrix $P$ and a diagonal matrix $D$ such that $P^{-1}AP = D$.

*Solution.* We first calculate the characteristic polynomial $p_A$.

$$\begin{aligned} p_A(x) &= \det(xI - A) \\ &= \det \begin{bmatrix} x-2 & 1 & 1 \\ 1 & x-2 & 1 \\ 1 & 1 & x-2 \end{bmatrix} \\ &= (x-2)^3 + 1 + 1 - (x-2) - (x-2) - (x-2) \\ &= x^3 - 6x^2 + 9x \\ &= x(x-3)^2. \end{aligned}$$

Thus, the eigenvalues are 0 and 3.

Next we compute the eigenspace corresponding to 0. This amounts to solving $(xI - A)v = 0$ when $x = 0$ which we do by elimination.

$$\begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1/2 & -1/2 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1/2 & -1/2 \\ 0 & -3/2 & 3/2 \\ 0 & 3/2 & -3/2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1/2 & -1/2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Thus, $v$ is an eigenvector for $A$ corresponding to 0 if and only if $v \neq 0$ and

$$v_1 - 1/2v_2 - 1/2v_3 = 0 \quad \text{and} \quad v_2 - v_3 = 0.$$

In particular,

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \text{ is an eigenvector for } A \text{ corresponding to } 0. \tag{1}$$

Next we compute the eigenspace corresponding to 3. This amounts to solving $(xI - A)v = 0$ when $x = 3$ which we do by elimination. There is just one step!

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Therefore $v$ is in the eigenspace corresponding to 3 if and only if

$$v_1 + v_2 + v_3 = 0.$$

Consequently,

$$\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \text{ are linearly independent eigenvectors corresponding to } 0. \tag{2}$$

Combining (1) and (2) yields that $P^{-1}AP = D$ where

$$P = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \text{ and } D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$